Math 10A HW13

(1) False, an eigenvector by definition must be a non-zero vector!

(2) True (refer to worksheet)!

(3) True!

Suppose λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$. Then

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}.$$

(4) True (refer to worksheet)!

(5)

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} = A$

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 6 = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$\lambda = -1, 4$ eigenvalues

$\boxed{\lambda_1 = -1}$ $A + I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \to \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 3 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ basis for $\lambda_1 = -1$: $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\boxed{\lambda_2 = 4}$ $A - 4I = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \to \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 3 & -2 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cc|c} -3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ basis for $\lambda_2 = 4$: $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} = A$

$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 0 & -1-\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda) - 0 = 0$

$\Rightarrow -2 - 2\lambda + \lambda + \lambda^2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$ eigenvalues

$\boxed{\lambda_1 = 2}$ $A - 2I = \begin{pmatrix} 0 & 3 \\ 0 & -3 \end{pmatrix} \to \left(\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -3 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ basis for $\lambda_1 = 2$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\boxed{\lambda_2 = -1}$ $A + I = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} \to \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ basis for $\lambda_2 = -1$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(6) $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{pmatrix}$

$= (3-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ -1 & 3-\lambda \end{vmatrix}$

$= (3-\lambda)[(2-\lambda)(3-\lambda) - 1] + [-(3-\lambda)]$

$= (3-\lambda)[6 - 2\lambda - 3\lambda + \lambda^2 - 1] + [\lambda - 3]$

$= (3-\lambda)[6 - 5\lambda + \lambda^2 - 1] + \lambda - 3$

$\lambda + \lambda$

$= 18 - 15\lambda + 3\lambda^2 - 3 - 6\lambda + 5\lambda^2 - \lambda^3 + \quad - 3$

$-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$

$\lambda = 1, 3, 4$ eigenvalues

(7) $\vec{y}' = \begin{pmatrix} 1 & 2 \\ 6 & -3 \end{pmatrix} \vec{y}$, $\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 6 & -3-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(-3-\lambda) - 12 = 0$$
$$-3 - \lambda + 3\lambda + \lambda^2 - 12 = 0$$
$$\lambda^2 + 2\lambda - 15 = 0$$
$$(\lambda + 5)(\lambda - 3) = 0$$
$$\lambda = 3, -5$$

$$\swarrow \qquad \searrow$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

<u>general sol'tn</u> : $\vec{y} = c_1 e^{3t} \vec{v}_1 + c_2 e^{-5t} \vec{v}_2$

$$= c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{3t} - c_2 e^{-5t} \\ c_1 e^{3t} + 3c_2 e^{-5t} \end{pmatrix}$$

<u>using initial conditions</u> :

$c_1 - c_2 = 1 \Rightarrow$ $c_1 = 1 + c_2$

$c_1 + 3c_2 = -1 \Rightarrow$

$1 + c_2 + 3c_2 = -1$

$4c_2 = -2$

$c_2 = -1/2$, $c_1 = 1/2$

So solution is $\vec{y} = \begin{pmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-5t} \\ \frac{1}{2} e^{3t} - \frac{3}{2} e^{-5t} \end{pmatrix}$

(8) $\vec{y}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \vec{y}$

$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{pmatrix} = 0$

$$\Rightarrow \quad (1-\lambda)(1-\lambda) - 4 = 0$$
$$1 - \lambda - \lambda + \lambda^2 - 4 = 0$$
$$\lambda^2 - 2\lambda - 3 = 0$$
$$(\lambda - 3)(\lambda + 1) = 0$$
$$\lambda = -1, 3$$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

general soltn : $\vec{y} = c_1 e^{-t} \vec{v}_1 + c_2 e^{3t} \vec{v}_2$

$$= c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -c_1 e^{-t} + c_2 e^{3t} \\ c_1 e^{-t} + c_2 e^{3t} \end{pmatrix}$$

(9)

$$A\vec{y}_1 = \begin{pmatrix} 7 & -2 \\ 15 & -4 \end{pmatrix}\begin{pmatrix} 2e^{2t} \\ 5e^{2t} \end{pmatrix} = \begin{pmatrix} 14e^{2t} - 10e^{2t} \\ 30e^{2t} - 20e^{2t} \end{pmatrix}$$

$$= \begin{pmatrix} 4e^{2t} \\ 10e^{2t} \end{pmatrix} \checkmark \quad (\vec{y}_1' = 2\vec{y}_1)$$

$$A\vec{y}_2 = \begin{pmatrix} 7 & -2 \\ 15 & -4 \end{pmatrix}\begin{pmatrix} e^{t} \\ 3e^{t} \end{pmatrix} = \begin{pmatrix} 7e^{t} - 6e^{t} \\ 15e^{t} - 12e^{t} \end{pmatrix} = \begin{pmatrix} e^{t} \\ 3e^{t} \end{pmatrix} \checkmark$$

$$(\vec{y}_2' = \vec{y}_2)$$

<u>eigenvalues</u>: 1 , 2

<u>eigenvectors</u>: $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

(10) eigenvalues: -3 , 1

eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(11) $a_n = a_{n-1} + 2a_{n-2}$

(a) $\lambda^2 = \lambda + 2$ equivalently $\lambda^2 - \lambda - 2 = 0$

(b) $A\vec{v}_{n-1} = \vec{v}_n$

$$A\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

Let $A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$. Then notice:

$$\alpha_1 a_{n-2} + \alpha_2 a_{n-1} = a_{n-1} \Rightarrow \alpha_1 = 0, \alpha_2 = 1$$

$$\alpha_3 a_{n-2} + \alpha_4 a_{n-1} = a_n \Rightarrow \alpha_3 = 2, \alpha_4 = 1$$

So $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$.

(c) $\det(A - \lambda I) = \det\begin{pmatrix} 0-\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix} = 0$

$\Rightarrow (0-\lambda)(1-\lambda) - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$

same as (a)!

(12) $y'' - y' - 2y = 0$

(a)

$r^2 - r - 2 = 0$ characteristic polynomial

$(r-2)(r+1) = 0$

$r = -1, 2$

$\Rightarrow$ general soln: $y(t) = c_1 e^{-t} + c_2 e^{2t}$

(b) $\vec{y}' = A\vec{y}$

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix}$$

$\alpha_1 y + \alpha_2 y' = y' \Rightarrow \alpha_1 = 0, \alpha_2 = 1$

$\alpha_3 y + \alpha_4 y' = y'' \Rightarrow \alpha_3 = 2, \alpha_4 = 1$

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

(c) $\det(A - \lambda I) = \det\begin{pmatrix} 0-\lambda & 1 \\ 2 & 1-\lambda \end{pmatrix} = 0$

$(0-\lambda)(1-\lambda) - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$

same as (a)!

(d) $\vec{y}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\vec{y}$

from (c) we have eigenvalues $-1, 2$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

general soln't'n : $\vec{y} = c_1 e^{-t}\vec{v}_1 + c_2 e^{2t}\vec{v}_2$

$$= c_1 e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}1\\2\end{pmatrix}$$

(13) True!

A $m\times n \Rightarrow A^T$ $n\times m \Rightarrow A^T A \Rightarrow n\times n$ result
$\underset{(n\times m)(m\times n)}{}$

(14) True!

(15) $A = \begin{pmatrix} 7.4 & 1 \\ 5.1 & 1 \\ 6.9 & 1 \\ 7.2 & 1 \\ 1.4 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 3.7 \\ 2.6 \\ 3.4 \\ 3.6 \\ 0.7 \end{pmatrix}$

(a)

$A^T A = \begin{pmatrix} 182.18 & 28 \\ 28 & 5 \end{pmatrix}$, $(A^T A)^{-1} = \begin{pmatrix} 0.0394 & -0.22 \\ -0.22 & 1.435 \end{pmatrix}$

$A^T b = \begin{pmatrix} 91 \\ 14 \end{pmatrix}$

$\begin{pmatrix} a \\ b \end{pmatrix} = (A^T A)^{-1} A^T b = \begin{pmatrix} 0.0394 & -0.22 \\ -0.22 & 1.435 \end{pmatrix}\begin{pmatrix} 91 \\ 14 \end{pmatrix} \approx \begin{pmatrix} 0.505 \\ 0.07 \end{pmatrix}$

$y = 0.505x + 0.07$

(b)

(c) $0.505(6) + 0.07 = 3.1$